

DIRECT STOCHASTIC EQUIVALENT LINEARIZATION OF SDOF SMART MECHANICAL SYSTEM

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Abstract

The first and second moments of response variables for SDOF system with pseudoelastic material are obtained by a direct linearization procedure. This procedure is an adaptation of well-known statistical linearization methods, and provides concise, model-independent linearization coefficients. The method can be applied to systems that incorporate any SMA hysteresis model having a differential constitutive equation, and can be used for zero and non-zero mean random vibration. This implementation eliminates the effort of deriving linearization coefficients for new SMA hysteresis model. In this paper the complete statistical response of SDOF system containing a mass and a bar made of SMA is obtained via direct linearization procedure. The model considered is modification of phenomenological one-dimensional constitutive model originally proposed by Graesser and Cozzarelli, which provides the capability to model both the martensitic twinning hysteresis and martensitic-austenite pseudoelastic behavior, typical of shape memory alloys. Response statistics for zero mean random vibration are obtained. Furthermore, non-zero mean analysis of the system is carried out and comparisons are made with Monte Carlo simulation.

Introduction

The analysis of mechanical systems under intense random (i.e., earthquake) excitation, requires the incorporation of nonlinear stress-strain relationships into solution methods for random vibration. Exact solutions for the response statistics of nonlinear systems are very limited, particularly when the material behavior is hysteretic, having a multi-value forced-deformation pattern with non-conservative energy dissipation. The lack of closed form solutions for hysteretic random oscillators necessitates the use of accurate methods for approximate solution.

Since the first random vibration study of an inelastic system, performed by Caughey (1960), many researches have applied various approximation methods to nonlinear stochastic analysis. These include amongst others, moment closure techniques (Iyengar *et al.*, 1978; Crandall 1980; Noori and Davoodi, 1990), stochastic averaging (Roberts 1986; Roberts and Spanos, 1986), equivalent nonlinear systems (Nielsen *et al.*, 1990), and an energy dissipation balancing procedure (Cai and Lin, 1990). While these methods offer alternative means for the analysis of hysteretic or generally nonlinear systems, the most widely published of approximation methods has been that of Equivalent Linearization. Formulations of the linearization procedure can be found in the literature (Caughey 1963; Iwan 1973; Spanos 1981). Equivalent Linearization has been used to analyze SDOF systems using several differential mathematical models for hysteresis with approximated solutions comparing favorably to results obtained by pure Monte Carlo Simulation. While the procedure of linearization for each of these models has been identical, following the format presented by Atalik and Utku (1976), each unique constitutive equation has required a unique analytical derivation for the equivalent linear system parameters. Based upon examination of the

derivations associated with these models, it is reasonable to conclude that as hysteresis models have evolve into more sophisticated analytical forms to describe progressively more generalized stress-strain behavior, the analytical effort of linearization has become increasingly complex and tedious.

The focus of this paper is to present a procedure for obtaining the response statistics (first and second moments) of a SODF system which incorporate a pseudoelastic shape memory alloy, that is generally applicable to any rate-type SMA constitutive model. The expressions for equivalent linear system parameters will be posed in a form that is convenient for numerical solution, so as to allow for a fully automated linearization process, to be accomplished by computer code.

A Modified Shape Memory Alloy Constitutive Model

Since SMA is intended to be used over a wide range of strain, the Graesser and Cozzarelli model (Graesser and Cozzarelli, 1991) is extended to represent the hardening of the SMA after the transition to martensite is completed as it has been suggested in the literature (Wilde *et al.*, 2000). As the load increases, the pure martensite follows the elastic response with modulus E_m . The modified model is of the form:

$$\begin{aligned} \dot{\sigma} = E \left[\dot{\varepsilon} - |\dot{\varepsilon}| \left(\frac{\sigma - \beta}{Y} \right)^{n-1} \right] u_I(\varepsilon) + E_m \dot{\varepsilon} u_{II}(\varepsilon) \\ + [3a_1 \dot{\varepsilon} \varepsilon^2 + 2a_2 \text{sign}(\varepsilon) \varepsilon \dot{\varepsilon} + a_3 \dot{\varepsilon}] u_{III}(\varepsilon) \quad \beta = E\alpha \left[\varepsilon - \frac{\sigma}{E} + f_t |\varepsilon|^c \text{erf}(a\varepsilon) \right] \end{aligned} \quad (1a,b)$$

where the functions $u_I(\varepsilon)$, $u_{II}(\varepsilon)$ and $u_{III}(\varepsilon)$ are defined as:

$$u_I(\varepsilon) = [1 - u_{II}(\varepsilon) - u_{III}(\varepsilon)] \quad u_{II}(\varepsilon) = \begin{cases} 1 & |\varepsilon| \geq \varepsilon_m \\ 0 & \text{otherwise} \end{cases} \quad (2a,b)$$

$$u_{III}(\varepsilon) = \begin{cases} 1, & \varepsilon \dot{\varepsilon} > 0 \text{ and } \varepsilon_1 < |\varepsilon| < \varepsilon_m \\ 0, & \text{otherwise} \end{cases} \quad (2c)$$

As it has been pointed out by Wilde the terms $E_m \dot{\varepsilon} u_{II}(\varepsilon)$ in Eq. (1a) describes the elastic behavior of martensite, which is activated when the strain is higher than ε_m . Strain, ε_m , defines the point when the transformation of SMA from austenite to martensite is completed. The smooth transition from the curve of slope E_y to slope E_m is obtained by adding the last term in Eq. (1a) that is evaluated only during loading and for strain $\varepsilon_1 < |\varepsilon| < \varepsilon_m$. The constant a_1 , a_2 and a_3 control the curvature of the transition. Those constants are selected so that the slopes of the function defined by the last term at points ε_1 and ε_m are consistent with the slopes of SMA “plastic” behavior and martensite elastic response. The smoothness of the transition is governed by the selection of slope at strain ε_2 . Figure 1 explains the introduced constants in a graphical form.

System Modeling

Let consider a simpler mechanical system shown in Figure 2. The length and the cross-section area of the bar are denoted by L and A . We assume that the mass m of the bar is

negligible. If x measures the displacement of the mass relative to the base and y measures the displacement of the frame, the governing equation of motion of the system is given by

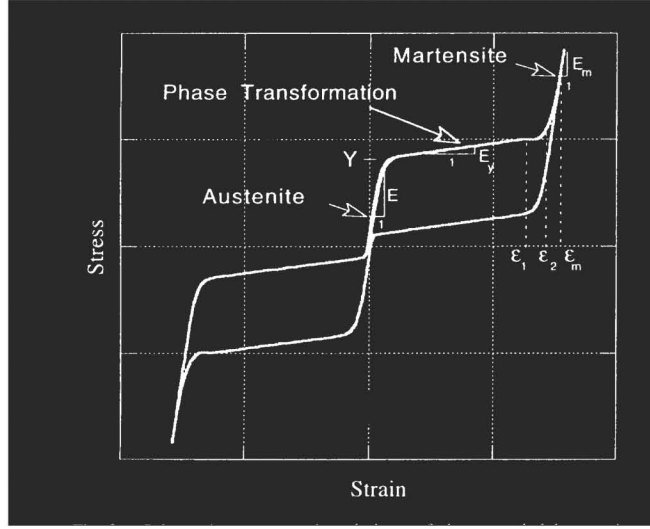


Figure 1. Schematic stress-strain relations of the extended hysteretic model of SMA (after Wilde *et. al.*, 2000)

$$m\ddot{x} + C\dot{x} + kx + A\sigma = mF(t) \quad (3)$$

where double dots denotes the derivative respect to time. Here a dashpot has been added to account for the viscous damping associated with metals in general which is not accounted for in the constitutive model. Following previous work (Feng and Li 1996) by using the bar length L as the characteristic length and $1/\omega$ as the characteristic time where ω is the dimensional natural frequency of the system we can non-dimensionalize Eq. (4).

$$\ddot{\varepsilon} + 2\zeta\dot{\varepsilon} + \omega^2\varepsilon + \frac{\sigma}{E} = f(t) \quad \dot{\sigma} = g(\dot{\varepsilon}, \sigma) \quad (4)$$

Since the SMA constitutive model we are considering is rate independent, Eq. (4) is not affected by rescaling. Here $g(\varepsilon, \sigma)$ is a general nonlinear constitutive equation that defines the SMA hysteretic behavior.

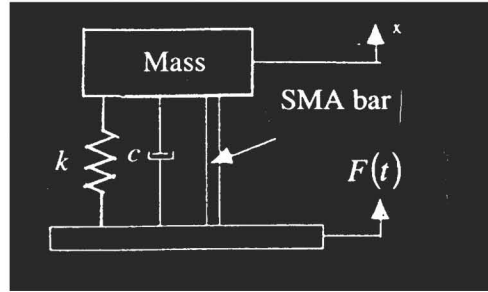


Figure 2. SDOF system containing a shape memory component

Statistical Equivalent Linearization

The method of equivalent linearization has been proven effective for providing first and second moments of the response for rate-type hysteretic systems. The procedure calls for

writing the system of Eqs. (4) in state-space format; letting $q_1 = \varepsilon$, $q_2 = \dot{\varepsilon}$ and $q_3 = \sigma$ be non-zero mean random variables, then from Eqs, (4):

$$\dot{q}_1 = q_2 \quad \dot{q}_2 = -\omega^2 q_1 - 2\zeta q_2 - \frac{q_3}{E} + F \quad \dot{q}_3 = g(q_2, q_3) \quad (5)$$

Taking the expected value of each equation above, with $\mu_i = q_i$ and $\mu_f = E[f(t)]$:

$$\dot{\mu}_1 = \mu_2 \quad \dot{\mu}_2 = -\omega^2 \mu_1 - 2\zeta \mu_2 - \frac{\mu_3}{E} + \mu_f \quad \dot{\mu}_3 = E[g(q_2, q_3)] \quad (6)$$

Subtracting Eqs. (6) from (5) yields zero-mean random variables, $y_i = q_i - \mu_i$ and $F = (f - \mu_f)$ such that:

$$\dot{y}_1 = y_2 \quad \dot{y}_2 = -\omega^2 y_1 - 2\zeta y_2 - \frac{y_3}{E} + F \quad \dot{y}_3 = g(q_2, q_3) - E[g(q_2, q_3)] \quad (7)$$

The nonlinear Eq. (7) involving the SMA constitutive hysteresis model $g(q_2, q_3)$ may be replaced with the linear form:

$$\dot{y}_3 = C_e y_2 + K_e y_3 \quad (8)$$

where C_e and K_e are unknown equivalent damping and stiffness parameters, respectively. This substitution allows for the generation of a linear system of ODE's whose solution provides response covariance,

$$S_{ij} = E[y_i y_j] \quad (9)$$

Equation (6) and (7) govern the means and covariance for the system response. They may be solved simultaneously for a suitable choice for C_e and K_e . Excitation of Gaussian White Noise simplifies $E[y_1 F] = E[y_3 F] = 0$, $E[y_2 F] = \pi W_0$, where W_0 is Power Spectral Density (PSD). Together, Eqs. (6) and (9) govern the means and covariance of system response. They may be solved numerically if a suitable choice for C_e and K_e can be made. For closure, C_e and K_e are chosen to minimize the error of the linear approximation in Eq. (7), in a statistical sense. Following the results of Atalik and Utku¹² under certain conditions, including the assumption that the state variables $\dot{\varepsilon}$ and σ are jointly Gaussian, the choice which minimizes error is given by the following:

$$C_e = E\left[\frac{\partial g(\dot{\varepsilon}, \sigma)}{\partial \dot{\varepsilon}}\right] \quad K_e = E\left[\frac{\partial g(\dot{\varepsilon}, \sigma)}{\partial \sigma}\right] \quad (10)$$

These equations for equivalent stiffness and damping coefficients are obtained from an expression that is applicable to MDOF systems with general (not necessarily hysteretic) nonlinearity. It should be noted that the values of these coefficients are system dependent, and must be continually updated as the system of ODE's for means and covariance's is solved numerically.

Direct Implementation of Stochastic Linearization

The hysteretic system of Eqs. (6) and (7) is an example of a system in which there is only one state variable that is governed by a nonlinear differential equation. This is an advantage to a system with multiple nonlinearities; and leads to the development of linearization

coefficients that differs from the definitions given in Eq. (10). Another issue that contributes to this development is the fact that although a linear system is used as an approximation to the equations governing means and covariance's, the response dependence of its linear system prevents its analytical solution. This system dependence of the linearization coefficients influences their new definition.

Consider the error of linearization; the difference between the nonlinear system and the approximation of Eq. (8):

$$\{e\} = \begin{Bmatrix} 0 \\ 0 \\ g - E(g) - C_e y_2 - K_e y_3 \end{Bmatrix} \quad (11)$$

Only the third variable is non-zero, therefore, the covariances of the error terms are all zero, except to the term:

$$E\{ee^T\} = E\left[(g - E(g) - C_e y_2 - K_e y_3)^2\right] \quad (12)$$

Since C_e and K_e are response dependent, and require continual re-evaluation in the numerical solution of Eq. (9), they may be treated as constants at each discrete time step, and brought outside the expected value operator. If $E[g]$ is assumed to be known at each time step, then $E[y_2 E[g]] = E[y_2] E[g] = 0$ and $E[y_3 E[g]] = E[y_3] E[g] = 0$, since y_2 and y_3 are zero-mean random variables. Nothing that $E[gy_2] = E[gq_2] - \mu_2 E[g]$ and $E[gy_3] = E[gq_3] - \mu_3 E[g]$, this lead us to the specific definition of linearization coefficients:

$$\begin{aligned} C_e &= \frac{S_6 E[g\dot{\epsilon}] - S_5 E[g\sigma] - S_6 E[g]\mu_2 + S_5 E[g]\mu_3}{S_4 S_6 - S_5^2} \\ K_e &= \frac{S_4 E[g\sigma] - S_5 E[g\dot{\epsilon}] - S_4 E[g]\mu_3 + S_5 E[g]\mu_2}{S_4 S_6 - S_5^2} \end{aligned} \quad (13)$$

In essence, both the classical and the currently method of linearization are founded upon the same principle, the mean square minimization of expected error. The difference is in the way the process is implemented, with the current method being better suited for computational methods. The advantage to the above definition over that of Eq. (10) and (14) is that this is posed directly in terms of the desired response statistics and only three expected values, which also do not contain partial derivatives of the SMA hysteresis model. With the excitation being Gaussian white noise, $\dot{\epsilon}$ and σ can be assumed to have a jointly Gaussian probability density function, thereby allowing the evaluation of the three unknown expected values $E[g]$, $E[g\dot{\epsilon}]$ and $E[g\sigma]$, and closure of the system. Using the mean and covariance response solutions of Eqs. (6) and (9), the expected values may be obtained through Gaussian-Quadrature integration. Whatever the choice of SMA hysteresis model, Eqs. (6), (9) and (13) remain unchanged, and the effort of linearization is reduced to simply changing the function of $g(\epsilon, \sigma)$ in the critical expected values. Often, this redefinition, or introduction of a new SMA model, requires the altering of only one line of computer code, and most importantly, completely avoids the type of analytic effort previously associated with equivalent linearization.

Conclusions

A procedure has been presented for obtaining the first and second moments of response variables for a SDOF system with a pseudoelastic component. The method is an adaptation of classical linearization procedures and eliminates the effort of analytical derivations associated with each SMA hysteresis model. Case of study using a modification to the well-known Graesser and Cozzarelli SMA constitutive model has been presented. The method is accurate when judge by Monte Carlo results, and possibly more accurate then classical linearization routines, due to reduced numerical error. It is possible that with the analytical effort removed from the task of random vibration analysis, this generally applicable procedure may promote the introduction of more sophisticated SMA hysteresis models.

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